[13.26] Express the coefficients of the polynomial



in diagrammatic form. Work them out for *n* = 2 and *n* = 3.

Proof. Beckmann produced a very nice proof that was then further simplified by an elegant enhancement provided by Dean. However, as far as I can tell, neither of them actually “worked out the equations for *n* = 2 or *n* = 3” as Penrose requested to generate the polynomial .

Let **R** = , **T** = , and **S** = = -** and set

** = 

Recall that  .

We use the following fact repeatedly:

:

Proof: = 

= 

✔

***n* = 2**:

2! Det (**T** –  **I**) = **

 ** * *

* *2 .

There is a basis such that the matrix of T is triangular, so that



(This is the Jordan Canonical Form. Penrose mentions it in Footnote 13.12.)

I will use the fact that  which I proved in problem [13.22]. So,

*c d*

*a b*















.

Similarly we find that

.

Finally,

= 2! = 2.

So

 ✔

***n* = 3**:

*d e f*

*a b c*

 = = – **

= – ** – ** + **2

= – ** – ** + **2

– ** + **2 + **2 – **3







+ …

.

(There are 36 sets of expressions involving T and ** corresponding to 6 permutations of  times 6 permutations of .)

By choosing an appropriate basis we can assume *T* is in triangular form:



We also use the following fact that I proved in problem [13.22]:



In any of the 36 sets of expressions, unless all three of the upper and lower indices match, the given set consists of the sum of eight zeros:

* Each of the 8 terms has at least one factor of ** or *T* from the lower left of its matrix. Those values are zero.

So, there are only 6 sets that have non-zero terms, and we can write















.

Therefore



. ✔